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Journal of the European Ceramic Society 32 (2012) 251–255
www.elsevier.com/locate/jeurceramsoc

Short Communication

Influence of measurement uncertainties on the determination of the Weibull distribution

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Received 25 July 2011; received in revised form 5 September 2011; accepted 12 September 2011

Abstract

The influence of measurement uncertainties (MU) on the determination of the parameters of a distribution function has been analyzed using a Monte Carlo simulation technique on the example of the Weibull distribution, which describes the strength of brittle materials. It is shown that in the parameter range which is relevant for strength testing of brittle materials (e.g. ceramics) very high precision measurements are necessary if the Weibull modulus of the parent distribution is $m \geq 20$. In that case the MU should be lower than ±2% of the measured value to obtain the same confidence level compared to the ideal case (MU = 0). Otherwise a significant underestimation of m becomes very probable. However, if $m \leq 10$, even relatively large MU (up to ±10%) are tolerable. In summary, the precision of the measurements is acceptable as long as the width of the error distribution be much smaller than the width of the parent distribution.

Keywords: Strength; Failure probability; Failure analysis; Test methods; Weibull statistics

1. Introduction

In many cases a physical property is not simply given by a single value, but has to be described by a distribution function. This distribution function is determined by many individual experiments. If the type of the distribution function is known, the number of experiments can be reduced, because only a few parameters (in many cases a scale and a shape parameter) and not the distribution function in each detail must be determined. In the case of the Gaussian and Poisson distributions, the parameters are the mean value and the standard deviation. As for the case of the Weibull distribution, the parameters are the characteristic strength and the Weibull modulus.

It is obvious that even the determination of a few parameters can cause a significant effort. If, for example, we select five random specimens (the sample size $N = 5$) to determine a Gaussian distribution, all the values of these five specimens can (randomly) be higher or lower than the mean of the distribution. As a consequence, both an incorrect mean and width of the distribution might be determined. It is also important to realize that, in almost any case, the properties of a sample are different to the properties of the parent distribution. Indeed the deviation tends to be smaller if the sample size increases (if, in the above example, the sample size were $N = 1000$, it would be much more unlikely that all properties of these 1000 specimens were higher than the mean of the parent distribution as it may be the case for $N = 5$). The consequence of this behaviour is that the parameters of a distribution cannot be precisely determined with any sample. Nevertheless, it is possible to define confidence intervals. For example a 90% confidence interval for the determination of the size parameter means that the size parameter of the parent distribution is, with a probability of 90%, within this range. Confidence intervals in dependence of the sample size can be found in literature.[1–3]

The above analysis is made for the hypothetical case that no measurement uncertainties (MU) occur. However, in the case of real measurements, MU always exist. In this paper we want to demonstrate how the determination of size and shape parameters of a distribution can be corrupted by MU. Namely, the range of the confidence interval and its confidence level become incorrect. The analysis is done on the example of strength measurements of brittle materials and is relevant for the daily testing

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doi:10.1016/j.jeurceramsoc.2011.09.008

practice of ceramics. In this case the data are Weibull distributed and large MU may occur during testing. The results can qualitatively be extended to any other distribution.

2. Strength of brittle materials

The tensile strength of brittle materials (e.g. ceramics or glasses) is determined by small defects, which are distributed within the specimens or on their surface. The strength of a specimen is determined by the most critical flaw (which is, in general, the largest one in the most highly stressed location). Its size differs from specimen to specimen. Therefore, the strength has to be described by a distribution function, which reflects the size distribution of flaws in the specimens. A wide flaw distribution causes large inherent scatter of strength. A consequence is the size effect of strength.[4–7] In large specimens, large flaws are more likely to occur than in small specimens and, therefore, the mean strength of a set of large specimens is expected to be lower than the mean strength of a set of small specimens. This behaviour is well described by the Weibull distribution:

$$F = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right] \qquad (1)$$

where F is the probability of failure, V is the specimen volume, σ is the applied stress, V_0 is a normalizing volume, σ_0 is the characteristic strength (the scaling parameter) and m is the Weibull modulus, which describes the scatter of the data (the shape parameter). Small values of m denote large scatter in strength. Traditional ceramics typically have m values about 5 or lower. However, for modern technical ceramics m values between 10 and 20 and even higher have been reported.[7,8]

Strength measurement techniques for brittle materials may suffer from large MU, which can bias the determination of the Weibull parameters (σ_0 and m). For standard bending testing (i.e. ASTM C1161, EN-834-1) of specimens with dimensions $3 \times 4 \times 45$ mm the MU have been analyzed in.[9] They are claimed to be less than ±2%. In many cases of practical relevance, bending test specimens have to be cut out of components (the flaw population is related to the processing procedure). For small components the standardized specimen size may be too big and smaller specimens have to be used. Of course MU increase with decreasing specimen size.[10,11] For instance, MU in bending specimens having a length of 15 mm may reach up to ±5%, even if all recommendations in standards are taken into account.

In many cases (e.g. if components are discs or plates or if the in service loading is biaxial) strength testing is done under biaxial conditions, e.g. using the ring on ring or the ball on ring test.[12] Here a clear full analysis of MU is missing. Indeed, some unevenness or roughness of specimen surfaces, e.g. when testing "as-sintered" specimens or components, causes an undefined load transfer. This may have a strong influence on the applied bending moments. Sliding friction occurs by pulling the specimen over the supporting ring during loading which is not taken into account. The curvature of specimens can also influence the load transfer from specimen to specimen. Therefore these tests can be very imprecise. In the case of biaxial tests the MU associated with the failure to align the supporting and loading rings can reach 10%[13] and these MU have to be considered when analyzing strength test results. In the ball-on-three-balls test developed recently by some of the authors (a biaxial tensile strength test) the MU is very low (±1% or less).[14–16] This high precision can be necessary to determine Weibull parameters of highly reliable materials, as it will be shown below.

3. Monte Carlo simulations

In previous work of some of the authors a Monte Carlo technique to simulate Weibull distributions was developed. A Weibull distribution is characterized by the Weibull parameters σ_0 and m (for simplicity we select $V = V_0$ in Eq. (1)). Thus, the probability of failure, F_i of the test i can be simulated dividing by a number between 0 and 1. The corresponding strength, σ_i, can be read from Eq. (1). By repeating this procedure N-times a sample of N strength tests is generated. From this sample the Weibull parameters can be determined in the usual way described in standards using the maximum likelihood method (see ASTM 1239-95[17] or EN-843-5[2]). By comparing these (virtual) parameters with the real ones (remember: the sample is always different to the parent distribution) the confidence intervals corresponding to the Weibull parameters as a function of N can be determined.[3]

In the actual study we started with a virtual Weibull distributed sample of N specimens, for a given σ_0 and m. We select $\sigma_0 = 1$ for simplicity. We further assume that the MU (X_i) are homogeneously distributed in an interval which is symmetrically arranged around the characteristic strength (σ_0) of the material (other distributions of MU, e.g. a Gaussian distribution, could also be assumed, but this would not change the trends of the results):

$$\left[\frac{\sigma_i - X_{\mathrm{MU,max}}}{\sigma_0}, \frac{\sigma_i + X_{\mathrm{MU,max}}}{\sigma_0}\right] \text{ or } [1 - x_{\max}, 1 + x_{\max}] \qquad (2)$$

where $X_{\mathrm{MU,max}}$ and $x_{\max}$ are the largest possible absolute and relative errors, respectively. We add, to each individual strength value σ_i of the virtual sample, an error, which is diced out of the interval given by Eq. (2). Then, the Weibull parameters of this new sample (i.e., σ_s and m_s) are determined. They are the simulated parameters, which correspond to the measured parameters and, in general, they differ from the original values. In order to get a statistics concerning this difference, the procedure was repeated several times (10^5 times in the actual case) for each set of parameters (m and $x_{\max}$).

For this study, parameters have been selected, which are typical for strength and strength testing of ceramic materials. Here standards recommend to use samples with $N = 30$ or larger (in practice they are not larger for cost reasons). Therefore the main trends are analyzed for samples of that size. In general, the parameters are varied as follows: $m = \{5, 10, 20, 30\}$, $N = \{5, 10, 30, 100\}$ and $x_{\max} = \{0\%, 2\%, 5\%, 10\%\}$.

4. Results and discussion

Fig. 1 shows the results of 10^5 Monte Carlo simulations of samples of size $N = 30$ diced out of a Weibull distribution with

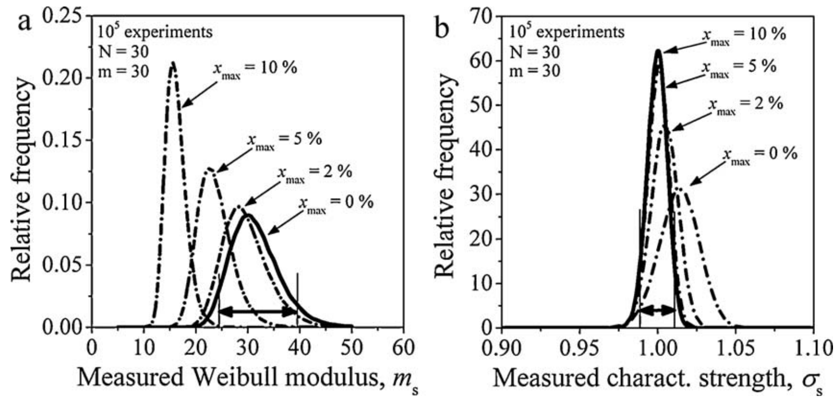

Fig. 1. Results of 10^5 Monte Carlo simulations of a Weibull distribution with $\sigma_0 = 1$ MPa and $m = 30$ (a) for the Weibull modulus and (b) for the characteristic strength. The sample size is $N = 30$. The full curves relate to simulations where no MU is assumed ($x_{max} = 0$). The other curves refer to simulations where the action of MU is taken into account. The width of the 90% confidence interval for the curves without MU are indicated by double arrows.

$\sigma_0 = 1$ MPa and $m = 30$ for several MU. Plotted is the relative frequency of the simulated (measured) Weibull modulus (Fig. 1a) and the characteristic strength (Fig. 1b). The full lines refer to the measurements where no MU occur and the broken lines to measurements where MU are considered.

For the curves without MU ($x_{max} = 0$) the most frequent measurement for the Weibull modulus is slightly larger than 30, but the mean is approximately $m_s = 32$, whereas the value of the parent distribution is $m = 30$. The Weibull distribution function is not symmetrical and therefore the data determined on samples (for the characteristic strength as well as for the modulus) are biased. Nevertheless, confidence intervals for the m_s and σ_s can be defined (the 90% confidence intervals are indicated by double arrows). The modulus and the characteristic strength of the parent distribution are within the confidence interval.[a]

The broken lines refer to simulations, where the action of MU is taken into account. With increase of the MU (x_{max}) it is more probable to measure a lower Weibull modulus ($m_s < m$), and a (slightly) higher characteristic strength ($\sigma_s > \sigma_0$) than the corresponding parameters of the parent distribution. Whereas the effect on the determination of the modulus is high, the effect on the characteristic strength is relatively weak. Therefore we will focus the following discussion on the modulus.

In Fig. 2 the measured values of the Weibull modulus (the mean value and the 90% confidence interval) are plotted versus the modulus of the parent distribution (m). As stated before, for $x_{max} = 0$ (Fig. 2a), there is a chance that the mean modulus is higher than the modulus of the parent distribution, but it is (for all values of the Weibull modulus) within the 90% confidence interval of the experimental data. For experiments with $x_{max} > 0$ (Figs. 2b–d) the experimentally determined values of the modulus become smaller than without MU. This effect is enhanced with increasing MU.

[a] In standards it is requested to correct measured values of the Weibull parameters by the "bias factor", since – in the mean – the "measured" parameters deviate from the real ones. Since the only correct statement is to say that the Weibull parameters of the parent distribution are – with the given probabilty (e.g. 90%) – within the corresponding confidence interval, the correction is in this case not necessary.

At $x_{max} = 2\%$ this effect is not very significant but it almost balances the bias mentioned above. However, at $x_{max} = 5\%$ and $x_{max} = 10\%$, it causes significant deviations of the measured values of the modulus from the real ones. This effect is even more pronounced at higher m values. At $x_{max} = 5\%$ and $m = 30$ the modulus of the parent distribution is just outside the 90% confidence interval of the measured values (i.e. the probability that the modulus of the parent distribution be within the experimentally determined confidence interval is only 5%) and for $x_{max} = 10\%$ the same applies for $m = 15$. It is clear that the MU corrupt the determination of the Weibull modulus. The nominal confidence interval as calculated on the basis of the error free measurements does no longer fit to the true measured data including MU.

In Fig. 3 the probability (y-axis) that the Weibull modulus of the parent distribution be within the nominal 90% confidence interval of the measurement is plotted versus the maximum relative error x_{max} (x-axis). The parameter in the curves is the Weibull modulus. To build up Fig. 3,the measured Weibull modulus, m_s, and its corresponding 90% confidence interval [$m_{s,\text{lower}}$, $m_{s,\text{upper}}$] have been calculated for every sample from the parent distribution (biased by a given x_{max}). Then, it has been checked whether m is within the measured interval. This has been accounted for all 10^5 samples. According to the chosen confidence interval of 90%, it is clear that for $x_{max} = 0$ the 90% of the samples measured (i.e. the 90% of the corresponding confidence intervals) will contain the parent parameter m. Thus, there is a 90% probability that the parent parameter m is within the 90% confidence interval of the measurement (this corresponds to the starting point of all curves in Fig. 3). Now, for materials with a low modulus ($m = 5$), no significant influence of the MU on the measurement can be appreciated in Fig. 3. This means that 90% of the m_s, calculated from each of the 10^5 samples (biased by a given MU), would contain the Weibull modulus of the parent distribution with a 90% probability. However, for materials with higher modulus (e.g. $m = 30$), a significant decrease of the quality of the measurement still occurs for small MU (e.g. at $x_{max} = 5\%$). In this case, for instance, only 35% of the measured samples will contain the modulus of the parent distribution with a "90% confidence interval". In other words, in two of three samples, the estimated Weibull modulus will not correspond to that

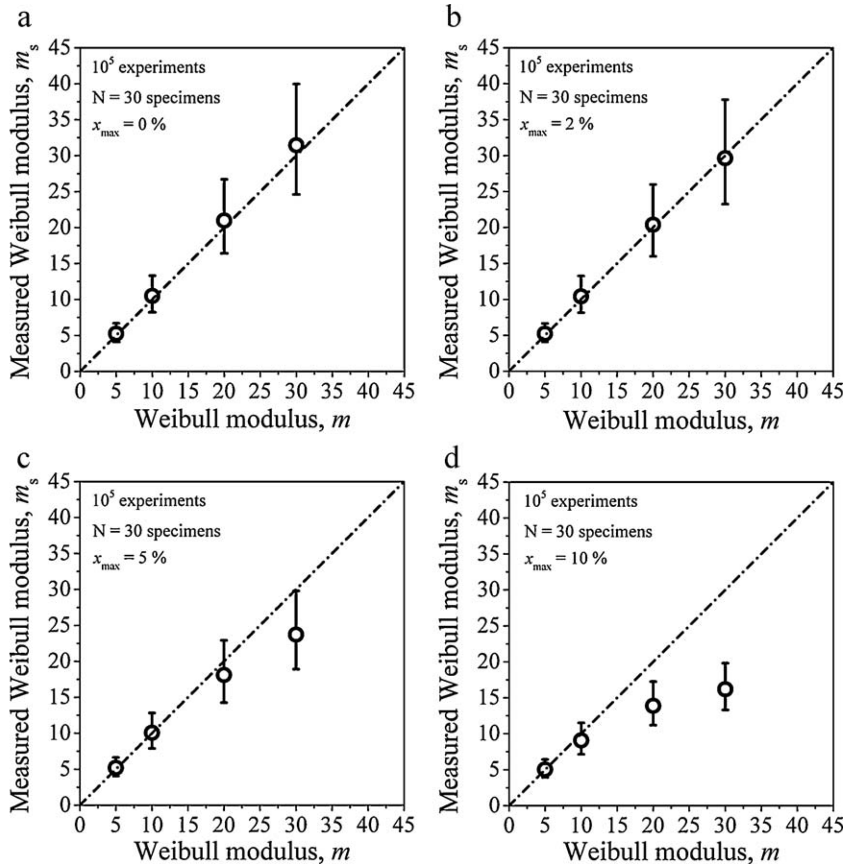

Fig. 2. Measured versus real Weibull modulus for samples of size $N = 30$ for experiments having a maximum relative MU of (a) 0%, (b) 2%, (c) 5% and (d) 10%. The symbols refer to the mean value of the experimental data. Also shown are the 90% confidence intervals.

of the parent distribution. For materials with an even narrower distribution (e.g. $m = 100$) and for $x_{\max} > 3\%$, the probability to measure the modulus of the parent distribution is almost zero.

The above analysis has been made for samples with $N = 30$. Indeed, the confidence intervals become wider if the sample size decreases and smaller if it increases, but this does not change the trends discussed above. For example, for $x_{\max} = 10\%$ and for $m = 30$, the mean value for m_s results in 17, 16 or 15.6 for $N = 10$, 30 or 100, respectively. Of course the influence of MU is less significant if the "measured" confidence intervals are wider.

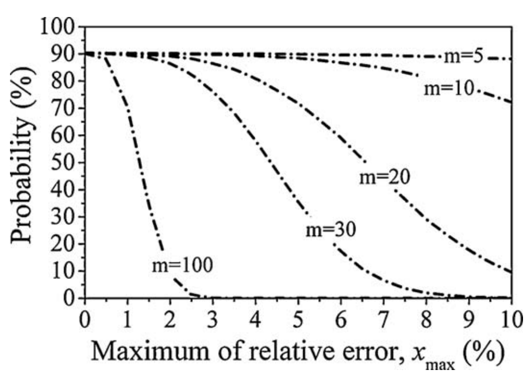

Fig. 3. Plotted is the probability that the Weibull modulus of the parent distribution is within the nominal 90% confidence interval of the measurement versus the maximum relative MU ($x_{\max}$) for samples of size $N = 30$. Plotted data are results from Monte Carlo simulations. Parameter in the curves is the Weibull modulus of the parent distribution.

In the experimental praxis MU are not always symmetrically distributed around the measured value. Very often there is a systematic (positive or negative) error in the measurement. This case has also been analyzed with Monte Carlo simulations in the parameter range defined above. Roughly speaking, a systematic error shifts the characteristic strength (the scale parameter) by the mean of this error $(X_{\mathrm{MU,\,max}} - X_{\mathrm{MU,min}})/2$. The influence on the Weibull modulus (the shape parameter) is almost the same as that of a symmetric MU distribution having the same width. In fact, if the systematic error increases the measured strength, the modulus increases, whereas if it decreases, the measured strength is lower compared to the action of a symmetric error, but this effect is not significant.

5. Conclusions

The influence of measurement uncertainties (MU) on the determination of the parameters of a distribution function has been analyzed using a Monte Carlo simulation technique on the basis of the Weibull distribution, which describes the strength of brittle materials. The sample is always different to the parent distribution. If the parameters of many (virtual) samples are assessed, confidence intervals can be determined, which reflect the probability of containing the parameters of the parent distribution. The larger the sample size, the smaller the confidence interval, thus increasing the probability of containing the parent parameter. However, in strength testing, the sample size is always relatively small, since the specimens cost is high. Therefore the effect of the sampling on the "measured" parameters of the distribution becomes highly relevant.

It follows that whereas the influence of MU on the size parameter (the characteristic strength) is relatively small (at least in the analyzed parameter range, which is typical for strength testing of brittle materials), the effect on the shape parameter (the Weibull modulus) may become very significant:

(i) For strength testing of brittle materials (ceramics) a high precision is necessary if the material has a Weibul modulus of $m \geq 20$. Then the maximum relative MU should be smaller than 5%. In bend testing, specimens longer than 30 mm are required. For the case of conventional biaxial strength testing, where relative MU up to 10% can be expected, the width of the error distribution may exceed the width of the measurement. Therefore, the distribution of the measured parameters cannot be determined. In this case, methods such as the ball on three balls test, where MU are less than 1%, are recommended.

(ii) For strength testing of conventional ceramic materials having a low Weibull modulus ($m \leq 10$), even unprecise measurement systems (with MU up to 10%) can, in principle, be tolerable for the correct determination of the strength distribution.

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